Imprimitivity Bimodules

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1 Motivation

Definition 1.1. Two rings R, S are *Morita equivalent* if the categories of left R-modules and left S-modules are equivalent.

In a series of seminal papers, Marc Rieffel developed a very useful notation of Morita equivalence for C*-algebras in the 70s.

2 Imprimitivity bimodules

Definition 2.1. Let A and B be C*-algebras. Then an A-B-*imprimitivity bimodule* (*A-B-equivalence bimodule*) is an A-B-bimodule such that

Actually (ii) $\Rightarrow$

$\operatorname{span}\{\langle x, y\rangle_A : x, y \in X\}$ dense in A

(i) X is a full left Hilbert A-module, and a full right Hilbert B-module;

$(a\cdot x)\cdot b = a\cdot(x\cdot b)$

(ii) for all $x, y \in \mathsf{X}$, $a \in A$, and $b \in B$,

$(\lambda a)\cdot(x\cdot b) = a\cdot(x\cdot(\lambda b))$

$$\langle a\cdot x, y\rangle_B = \langle x, a^*\cdot y\rangle_B \quad \text{and} \quad \langle x\cdot b, y\rangle_A = \langle x, y\cdot b^*\rangle_A;$$

↳ A acts on right Hilbert B-module X_B as adjointable operators.

$\varphi : A \to \mathcal{L}(X_B)$

(iii) for all $x, y, z \in \mathsf{X}$,

$$\langle x, y\rangle_A \cdot z = x\cdot\langle y, z\rangle_B.$$

Definition 2.2. Two C*-algebras A and B are *Morita equivalent* if there is an A-B-imprimitivity bimodule ${}_A\mathsf{X}_B$.

For a fixed $a \in A$

$X_B \longrightarrow X_B$

$x \longmapsto a\cdot x$

left A-module

${}_A\langle\cdot,\cdot\rangle$

Example 2.3. A Hilbert space $\mathcal{H}$ is a $\mathcal{K}(\mathcal{H})$-$\mathbb{C}$-imprimitivity bimodule, with

rank-one operators

Hilbert space is a Hilbert $\mathbb{C}$-module $\langle h,k\rangle_{\mathbb{C}} = \langle k,h\rangle$.

$$\langle h,k\rangle_{\mathbb{C}} := \langle k,h\rangle \quad \text{and} \quad \langle h,k\rangle_{\mathcal{K}(\mathcal{H})} := h\otimes \overline{k} = \Theta_{h,k} : \mathcal{H} \to \mathcal{H}$$

$\ell \mapsto \sum_{i=1}^{n} h\cdot\langle k,\ell\rangle_{\mathbb{C}}$ '' $\langle \ell,k\rangle h$

fullness $K(\mathcal{H})$-module

rank-one operators → finite rank operators $\xrightarrow{\text{closure}}$ compact operator.

Example 2.4. The C*-algebra A is an A-A-equivalence bimodule, with

$$_A\langle a,b\rangle = ab^* \quad \text{and} \quad \langle a,b\rangle_A = a^*b.$$

fullness : elements of the form a^*b is dense in A.

Example 2.5. Let A and B be isomorphic C*-algebra with $*$-isomorphisms $\varphi : A \to B$. One can construct an A-B-imprimitivity bimodule $_A\mathsf{X}_B$ with underlying space B by

$$x\cdot b = xb, \quad a\cdot x = \varphi(a)x, \quad \langle x,y\rangle_B = x^*y, \quad \text{and} \quad \langle x,y\rangle_A = \varphi^{-1}(xy^*).$$

Isomorphic $\implies$ Morita equivalent.

$M_n(A)$ - $M_m(A)$-impri. bimod.

$\bigoplus_{i=1}^{n} A$

Example 2.6. For any positive integer n, A^n is an $M_n(A)$-A-imprimitivity bimodule. For $u,v \in A^n$,

$u = \langle u_1, \dots, u_n\rangle$
$v = \langle v_1, \dots, v_n\rangle$

$$\langle u,v\rangle_A = \sum_{i=1}^{n} u_i^* v_i \quad \text{and} \quad \left(\langle u,v\rangle_{M_n(A)}\right)_{ij} = u_i v_j^*.$$

$pAp = \{pap \in A : a \in A\}$, $\overline{ApA} = \overline{\text{span}}\{apb : a,b \in A\}$

Example 2.7. Let p be a projection in A (or $M(A)$ when A is non-unital). Then Ap is an $\overline{ApA}$-pAp-equivalence bimodule, with

→ $\{ap : a \in A\}$

$$\langle ap,bp\rangle_{pAp} = pa^*bp \quad \textit{and} \quad \langle ap,bp\rangle_{\overline{ApA}} = apb.$$

3 Useful tools

Lemma 3.1. *Let A and B be C^*-algebras and suppose that X is an A-B-bimodule satisfying (i) and (iii) of Definition 2.1. Then X is an A-B-imprimitivity bimodule if and only if X satisfies (ii)' for all $a \in A$, $b \in B$, and $x \in \mathsf{X}$,*

$$\langle a \cdot x, a \cdot x \rangle_B \le \|a\|^2 \langle x, x \rangle_B \quad \textit{and} \quad \langle x \cdot b, x \cdot b \rangle_A \le \|b\|^2 \langle x, x \rangle_A$$

($\Rightarrow$) $\underline{T \in \mathcal{L}(X),}$

$\langle T(x), T(x) \rangle_A \le \|T\|_{op} \langle x, x \rangle_A$

(as elements of C^*-algs)

($\Leftarrow$) Fullness condition +

Cauchy-Schwarz ineq.

Proposition 3.2. *Every full Hilbert B-module X_B is a $\underline{\overset{A}{\overset{\shortparallel}{\mathcal{K}(\mathsf{X})}}}$-$B$-imprimitivity bimodule with* 'rank-one operators'

$$\langle x, y \rangle_{\mathcal{K}(\mathsf{X})} := \underline{\Theta_{x,y}}.$$

Conversely, if X is an A-B-imprimitivity bimodule, then there is an isomorphism ϕ of A onto $\mathcal{K}(\mathsf{X})$ such that $\phi(\langle x, y \rangle_A) = \langle x, y \rangle_{\mathcal{K}(\mathsf{X})}$ for all $x, y \in \mathsf{X}$.

Gives Alternative defn of Morita equivalence:

Take $X_B = \mathcal{H}_{\mathbb{C}}$

Definition 3.3. Let A and B be C*-algebras and $A_0 \subset A$ and $B_0 \subset B$ dense $*$-subalgebras. An A_0-B_0-*pre-imprimitivity bimodule* is a vector space X_0 which is an A_0-B_0-bimodule such that

missing $\langle x, x\rangle_A = 0 \Rightarrow x = 0$

(i) X_0 is a left pre-inner product A_0-module and a right pre-inner product B_0-module.

(ii) ${}_{A_0}\langle X_0, X_0\rangle$ and $\langle X_0, X_0\rangle_{A_0}$ span dense ideals of A and B, respectively.

$N = \{ x \in X_0 : \langle x, x\rangle_0 = 0 \}$

$X = \overline{X_0 / N}$

(iii) for all $a \in A_0$, $b \in B_0$, $x \in \mathsf{X}_0$,

$$\langle a{\cdot}x, a{\cdot}x\rangle_{B_0} \leq \|a\|^2 \langle x, x\rangle_{B_0} \quad \text{and} \quad \langle x{\cdot}b, x{\cdot}b\rangle_{A_0} \leq \|b\|^2 \langle x, x\rangle_{A_0}$$

in the completions B and A, respectively.

(iv) for all $x, y, z \in X_0$,

$$\langle x, y\rangle_{A_0} \cdot z = x \cdot \langle y, z\rangle_{B_0}.$$

Lemma 3.4. *Let A and B be C^*-algebras and $A_0 \subset A$ and $B_0 \subset B$ dense $*$-subalgebras. If X_0 is an A_0-B_0-pre-imprimitivity bimodule, then*

$$\|x\|_A^2 = \|\langle x, x\rangle_{A_0}\| = \|\langle x, x\rangle_{B_0}\| = \|x\|_B^2$$

for all $x \in \mathsf{X}$.

Punchline: An A_0-B_0-pre-imprimitivity bimodule X_0 can be completed to an A-B-imprimitivity bimodule X.

4 Induced representation of group C*-algebras

Recall: Suppose A acts as adjointable operators on a Hilbert B-module X_B. Given a nondegenerate representation $\pi : B \to \mathcal{B}(\mathcal{H}_\pi)$, we can

(i) form the interior tensor product of Hilbert C*-modules $\mathsf{X} \otimes_B \mathcal{H}_\pi$

Hilbert space

(ii) induce a representation $\operatorname{Ind}\pi$ of A on $\mathsf{X} \otimes_B \mathcal{H}_\pi$ via

$$\operatorname{Ind}\pi(a)\,(x \otimes_B h) := (a \cdot x) \otimes_B h.$$

Let G be a unimodular locally compact group, and H a closed (unimodular) subgroup of G. Last time, we constructed a right Hilbert $C^*(H)$-module $\mathsf{X}_{C^*(H)}$ (by completing $\mathsf{X}_0 = C_c(G)$), and we can induce representations of $C^*(H)$ to representations of $C^*(G)$ (equivalently, induce unitary representations of H to unitary representation of G).

Right $C^*(H)$ module structure: $\mathsf{X}_0 = C_c(G)$ is a right pre-inner product $C_c(H)$-module with

$$f \cdot b(s) = \int_H f\left(st^{-1}\right) b(t)dt$$

and

$$\langle f, g\rangle_{C_c(H)}(s) = \int_G \overline{f(r)}g(rs)dr,$$

for all $f, g \in C_c(G)$ and $b \in C_c(H)$.

$C^*(G)$ **acts as adjointable operators:** $z \in C_c(G)$ acts on $f \in X_0 = C_c(G)$ by

$$z \cdot f(s) = \int_G z(r)f\left(r^{-1}s\right)dr.$$

Extends this left action to a $*$-homomorphism of $C^*(G)$ into $\mathcal{L}\left(\mathsf{X}_{C^*(H)}\right)$. However, the image of this $*$-homomorphism is *not* $\mathcal{K}\left(\mathsf{X}_{C^*(H)}\right)$.

As it turns out, X is a $C_0(G/H) \rtimes G$-$C^*(H)$-imprimitivity bimodule, where $C_0(G/H) \rtimes G$ is the *crossed product C^*-algebra.*

$C^*(G) = \overline{C_c(G)}^{\text{universal}}$

Definition 4.1. $E_0 = C_c(G \times G/H)$ is a $*$-algebra with

$$\phi * \psi\,(r, sH) = \int_G \phi(u, sH)\psi\left(u^{-1}r, u^{-1}sH\right) du,$$

and

$$\phi^*(r, sH) = \overline{\phi\left(r^{-1}, r^{-1}sH\right)}.$$

The crossed product C*-algebra $C_0(G/H) \rtimes G$ is the completion of E_0 with respect to the norm

$\|\phi\| = \sup\{\|\pi \rtimes U(\phi)\| : (\pi, U)$ is a covariant representation of $(C_c(G/H), G)\}$

For any $\phi \in E_0$ and $f, g \in C_c(G)$, we define

$$\langle f, g\rangle\,(r, sH) = \int_H f(st)\overline{g\left(r^{-1}st\right)}dt$$

and

$$\phi \cdot f(s) = \int_G \phi\,(r, sH)\, f\left(r^{-1}s\right) dr.$$

Then $\mathsf{X}_0 = C_c(G)$ is a left pre-inner product E_0-module. Moreover, $\langle X_0, X_0\rangle_{E_0}$ spans a dense ideal in E_0 with respect to the sup (universal) norm defined above.

By completing the E_0-$C_c(H)$-pre-imprimitivity bimodule, we arrive at a $C_0(G/H) \rtimes G$-$C^*(H)$-imprimitivity bimodule.

Reference

[RW] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Mathematical Surveys and Monographs no. 60, American Mathematical Society, Providence RI, 1998.